# On Relativistic Quantum Mechanics

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## Abstract

A new way of factorizing the Klein-Gordon equation is proposed, which applies to fields of any type. The spin one-half case leads essentially to the Dirac equation. However, a new interpretation is given, in which the occurrence of anti-particles is a consequence of the two-valuedness of the representations of the Lorentz group in Hilbert space. Boosts have in these representations an anti-unitary component which leads to a change of the norm of wave functions. Under a boost, a state originally at rest develops a positive-frequency component. While the mass of a wave packet is a conserved quantity, its energy transforms according to the law of special relativity.

## Introduction

This note is motivated by the observation that many fundamental problems in Cosmology [1] might find a solution, if the concept of a negative gravitational mass is introduced [2].

The Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\psi(x) = 0 = (P^\mu P_\mu - m^2)\psi(x) \qquad (1)$$

governs the behavior wave functions, $\psi$ , in relativistic, non-interacting situations. It is invariant under inhomogeneous Lorentz transformations. It's abelian translational part implies the standard space-time dependence

$$e^{\pm i\, p_\mu x^\mu} \quad , \qquad (2)$$

a pair of independent solutions, which can be characterized by the sign of the zero component of the wave vector $p_0$. They are usually referred to as positive- and negative-frequency solutions.

For spin ½, Dirac achieved a full treatment by 'factorizing' the Klein-Gordon equation into two first-order equations. However, for bosons no analogous treatment has been made, because there seemed to be no need to do so.

In this note we propose an analogous procedure for systems of arbitrary spin, which draws from, and modifies the structure of Dirac's treatment.

## Factorizing the Klein-Gordon Equation

The Klein-Gordon equation is quadratic in the infinitesimal time translation operator

$$\partial_0 = -i\,P_0 \quad , \tag{3}$$

but one would like to have an equation linear in $P_0$ which would allow to obtain time dependence by a single integration. For a particle at rest this can be achieved directly and there are two solutions of positive and negative mass, which are the zero-momentum representatives of negative-and positive-frequency states, respectively. At this point we refer to the concept that time translation is provided by a mass operator with a spectrum unbound for positive as well as negative values [3, 4] (see also below).

To include moving particles we have first to consider these states in more detail. For the free states we have

$$_{+m}\psi(p,n) = e^{-i\,p^{\mu} x_{\mu}}\,u_n(p) \quad , \tag{4}$$

where $u_n(k)$ is a finite-dimensional spinor, which depends on the spatial components of $p$ alone, because we have $p^{\mu}p_{\mu} = 1$. Conventionally, the time translational operator (3) is considered to be the (positive-definite) energy operator such that $p_0$ is bound to be positive, whence the wave functions (4) are called negative-frequency states.

However, we have proposed that (3) is actually a mass operator with positive and negative eigenvalues, from which the energy operator is obtained by multiplying the mass eigenvalues with their sign [4]. Thus, henceforth we also call the negative-frequency states positive-mass states since we associate them with matter states. Positive-frequency states, which are conventionally discarded, are also called negative-mass states and are associated with anti-matter.

Negative mass states are obtained from (4) by applying the transformation $PT$, a simultaneous inversion of space and time, which is also a symmetry operation of Klein-Gordon equation (1). Obviously, it's representation in the complex state space is complex conjugation, $\bar{I}$, an anti-unitary operator. Now consider the anti-hermitean operator, $\vec{\rho}$, the vector of so-called infinitesimal rotations (actually basis elements of the Lie algebra of rotations) applied to the wave function. The scalar product $\vec{P}\vec{\rho}$ corresponds to a rotation about the direction of the vector $\vec{P}$ by an amount of it's norm. The operator $\bar{I}\,\vec{P}\vec{\rho}$, in addition, switches from positive- to negative-mass states and vice versa. Now we can setup the equation we are looking for:

$$(P^0 - \bar{I}\,\vec{P}\vec{\rho})\,\psi = m\,\tau\,\psi \quad \text{, with its adjoint} \quad (P_0 + \bar{I}\,\vec{P}\vec{\rho})\,\psi = m\,\tau\,\psi \quad . \tag{5}$$

Here, $\psi$ has a positive-mass component as well as a negative-mass one. With respect to this basis $P^0$ is multiplied with the (2 by 2) identity matrix, while $\tau$ is also diagonal but multiplies the states with the sign of the mass. In contrast, $\bar{I}$ has zero's on the diagonal and one's as off-diagonal elements.

Applying the operator and its adjoint yields

$$(P^0 - \bar{I}\,\vec{P}\vec{\rho})(P_0 + \bar{I}\,\vec{P}\vec{\rho})\,\psi = (P^0 P_0 - \bar{I}\,\vec{P}\vec{\rho}\,\bar{I}\,\vec{P}\vec{\rho})\,\psi = (P^0 P_0 - (\vec{P})^2)\,\psi = P^{\mu} P_{\mu}\,\psi = m^2\psi \quad , \tag{6}$$

which is the Klein-Gordon equation (1). The first equation uses the fact that the $p^{\mu}$'s commute among themselves, and with $\bar{I}$, because they are hermitean, and because $P^0$ commutes with $\vec{\rho}$. The last equation uses $\tau^2 = 1$. The second equation follows from two facts: firstly, one has $\vec{P}\vec{\rho} = \vec{\rho}\vec{P}$, because a translation commutes with a rotation about its translation axis; secondly, $\vec{\rho}$ is anti-hermitean such that $\vec{P}\vec{\rho}$ and $\bar{I}\,\vec{\rho}\vec{P}$ simply undo each others rotation. The remaining $\bar{I}^2$ yields the unit operator. Now, the need to add the operator $\bar{I}$ to the space-translational part in (5) becomes evident. It insures that the operation of equation (6) yields the relativistic invariant Klein-Gordon equation.

## Symmetry Considerations

The symmetry group of (1) is the Poincare group, consisting of translations in space and time and the invariant subgroup of Lorentz transformations. The plane wave states (2) are one-dimensional invariant states which, under a translation by a 4-vector *s*, are multiplied by the uni-modular factor

$$e^{-i\,p^{\mu}s_{\mu}} \quad . \tag{7}$$

That positive- and negative-frequency states combine owing to (5) poses no problem because this combination always involves the anti-unitary operation $\bar{I}$.

The behavior under Lorentz transformations is somewhat more involved. Let us first consider the analogous group *O*(4) of proper rotations in 4-dimensional space, which leave the 4-Euclidean norm invariant. In contrast to translations, rotations do not leave plane waves invariant but transform them among themselves, such that waves with wave-vectors of equal (Euclidean) norm are exchanged.

*O*(4) is a direct product of two groups which are each isomorphic to *O*(3) [5]. For the first one, the generators correspond to selecting the three distinct possible double-pairs of coordinate axes and to performing an infinitesimal rotation among the axes of the one pair combined with a rotation by the same amount among the remaining two axes of the second pair. To obtain the corresponding element of the second factor group, the rotation among the two remaining axes has to be in the opposite sense. Now, if we introduce circular parameters, $\varphi$, in these groups we see, that the effect on the above described transformations corresponds to rotations about only half of the value of the circular parameter [5]. Thus, considering the representation in Hilbert space, upon completion of a full circle in one of these factor groups we end up in a plane wave with inverted *k*-vector, or equivalently in the complex conjugate wave. This signifies that transformations in Hilbert space make up a two-valued representation of *O*(4).

Now, we look at two plane waves, *W*(*p*), with p-vectors along a given, say, the 0-direction ($p_0 = \pm 1$). There is the subgroup of rotations (in 1-2-3 space), the so-called little group, *L*, of *p*, which do not affect these *p*-vectors. Now consider a rotation, $R_L$, in an arbitrary plane in 1-2-3-subspace. This rotation can be combined with a rotation, $R_p$, among the orthogonal axis (within 1-2-3 space) to the chosen rotation plane (of $R_L$) and the *p* (i.e. 0) -axis. If the rotation angles in the two planes are equal or opposite this combination of rotations is exactly a member of one of the factor groups of *O*(4).Thus, simultaneous rotation by an angle of $\pi$, in the two planes, completes a full circle in either of the factor groups, which means that we are back to the unit operator of *O*(4). However, in the space of plane waves we end up with the complex conjugate, *W*(-*p*), of both waves, which is a purely anti-unitary operation. A rotation by $2\pi$, corresponding to two full circles in a factor group, restores the original waves and is thus the (unitary) unit-operation. Therefore, rotations by a different value lead to a combination of a unitary operator (diagonal part) and a anti-unitary one (off-diagonal part):

$$\cos(\frac{\Phi}{2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\bar{I}\sin(\frac{\Phi}{2})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad , \tag{8}$$

where $\varphi$ is the corresponding circular parameter in a factor group. Here, it is understood that the direction of *p* of the pair of waves follows the *O*(4) transformation in the usual way.

If the rotations in the two planes are by a different amount, we can always first perform a rotation in the chosen plane (in 1-2-3 space) by such an amount that the remaining angle is equal or opposite to the second-plane rotation-angle (by a partial rotation $R_L$). This is an element of the little group which leaves the *p*'s unchanged. What then remains is a pure transformation of the above described nature.

Now we can also see what happens in the case of Lorentz transformations. Formula (8) changes to

$$\cosh(w)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\bar{I}\sinh(w)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad . \tag{9}$$

The “rotation” $R_p$ involving the $p$ (i.e. 0) -axis now corresponds to a boost, cos and sin are replaced by cosh and sinh with an argument, $w$, now acyclic, which varies in the interval from minus to plus infinity, and which we can conveniently represent by

$$w=\frac{1}{2}\text{artanh}(v) \quad , \tag{10}$$

to make connection with the speed, $v$, with respect to the rest-system.

We call pure-boost transformations those, that also include a simultaneous rotation about the boost direction by an amount of $\pm\pi v/2$, such that a rotation by $\pm\pi/2$ is achieved when $v$ approaches the speed of light ($v \rightarrow 1$). Loosely speaking, these are the transformations in direction towards the *PT*-point in the space of Lorentz-transformations.

Now, a pure matter state at rest $W(l,0,0,0)$ , $W(-l,0,0,0) = 0$, is boosted (in 1-direction for simplicity) to

$$\cosh(w)\,W(l\cosh(w),l\sinh(w),0{,}0)+\sinh(w)\,W(-l\cosh(w),-l\sinh(w),0{,}0) \quad . \tag{11}$$

There are a few points to emphasize here:

- Representations of boosts mix positive- and negative-frequency states.
- Complex conjugation is a naturally occurring operation in Hilbert space, compare (5).
- The conventional norm, $N$, in Hilbert space is not conserved with boosts.
- Instead we have $N^2=\cosh^2(w)+\sinh^2(w)=\cosh(2\text{w})=\frac{1}{\sqrt{1-v^2}}=\gamma$ (12)

The last points are especially remarkable, because they lead to a new interpretation of the wave function. Consider the mass operator $M=P_0=i\partial_0$ which produces a value $p_0$ in the negative-frequency subspace and $-p_0$ in the positive-frequency one, such that its expectation value for a pure matter state (no positive frequency component at rest) wave packet is (appreciate the Lorentz contraction!)

$$\langle M\rangle=\frac{p_0}{\gamma}[\cosh^2(w)-\sinh^2(w)]=m \quad , \tag{13}$$

where we have used

$$p_0=m\gamma \quad . \tag{14}$$

This is precisely the rest mass $m$ of the particle, a scalar conserved under orthochronous Lorentz transformations. For an antiparticle (pure positive-frequency state at rest) the result is $<M> = -m$. Here we can also see the effect of splitting the proper Lorentz transformations into an orthochronous and a space-time reversing part which involves the space-time inversion (*PT*). The orthochronous transformations mix positive and negative-frequency components of the wave function but conserve the scalar rest-mass (including it's sign) which is expressed in the “Pythagorean”-formula of the hyperbolic functions in (13).

The energy is obtained by multiplying the contribution of the antimatter component by minus one [4], which yields

$$\langle E\rangle=\frac{p_0}{\gamma}(\cosh^2(w)+\sinh^2(w))=p_0=m\gamma \quad , \tag{15}$$

equally for matter and antimatter states.

The interpretation of the wave function is now such, that there is a matter- and an antimatter component. Under the acceleration of a boost, a particle originally at rest develops an anti-matter component, according to the above described transformation. One may object, that a boost just switches to a new inertial state, and there is no way to decide whether the particle has been accelerated. However, this is an incomplete view, because every mass at rest has associated with it a stationary gravitational field and an acceleration brings it locally out of equilibrium with that field.

Incidentally, this consideration clarifies (in our view) the so-called twin paradox [6] of special relativity because the differing world lines of two bodies (twins) can only be brought together more than once by intermediate acceleration. But acceleration decreases the length of a world line (local-time interval) [6] between the two crossing events, and we can distinguish the acceleration of each body.

# Particles with Internal Degrees of Freedom

Up to now we have just considered the case of a scalar particle. However, elementary particles may have internal degrees of freedom, which are described by an irreducible representation under Lorentz transformations. These representations are characterized by a pair of non-negative half-integers ($k$, $l$) and have dimension $(2k+1)(2l+1)$, while ($l$, $k$) is the complex-conjugated representation. Thus, when positive-mass states transform according to a ($k$, $l$) representation, the associated negative-mass states transform with a ($l$, $k$) representation, e.g. for spin ½ we have $k$ = ½, $l$ =0. For this latter example exponentiation of an infinitesimal pure boost is still rather simple owing to the properties of Pauli-matrices, $\sigma_i^2 = 1$, which yields

$$(\vec{v}\,\vec{\sigma})^2=v^2 \quad . \tag{16}$$

Transformation (9) becomes in this case

$$\cosh(w)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\bar{I}\sinh(w)\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix} \quad . \tag{17}$$

This may be compared with the conventional treatment (Dirac-spinor formalism, see e.g. [7]) which insists in having strictly negative-frequency states. This, at the cost of introducing particles and anti-particles as separate entities, the latter having unphysical negative energies.

Obviously, for higher-dimensional representations exponentiation of infinitesimal boosts becomes rather tedious and a thorough treatment is not within the scope of this short note.

# Zero-Mass Case

The zero-mass case is more delicate, because there is no direct analog in the $O(4)$ situation. Here, we restrict ourselves to a few remarks.

The time-translational operator must also have an unbounded spectrum. There must be positive and negative frequency states. However, there are no states at rest. Furthermore, it appears that no half-odd-integer (fermion) zero-mass fields are found in nature. It is an interesting question, whether this might be a consequence of the Lorentz symmetry of the wave equation (5).

Regarding negative-mass states of zero rest-mass systems, we may just mention, that the wave equation for photons has two types of solutions, in which the Poynting vector is either parallel or anti parallel to the wave vector. On the grounds that the Poynting vector describes the energy flow, the latter are conventionally dismissed as unphysical. However, if the Poynting vector is interpreted as mass flow, the energy flow of the negative-mass states points in the opposite direction to the Poynting vector and, hence, also in the direction of the wave vector. Thus, in the view presented here, anti-photons are to be expected in nature, though they would only be emitted or absorbed by anti-matter and not by matter. Consequently, one might speculate that the conspicuous cosmic voids, which fill about half the universe, may just be occupied by an anti-matter equivalent to the observed Cosmos [8].

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